SOLUTION OF A TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEM FOR A SECTOR

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We consider a problem for the Laplace equation in a circular sector wherein heat exchange takes place on the sides of the sector in accordance with Newton's law and a temperature distribution is specified on the circular arc.

1. In the plane of the complex variable w let D denote a circular sector of radius r and central angle π/m, where m is a positive integer:

$$D=\left\{w=r\exp(i\varphi): r\in(0,\ 1),\ \varphi\in\left(-\frac{\pi}{2m},\ \frac{\pi}{2m}\right)\right\}. \tag{1}$$

We can represent the boundary ∂D of domain D in the form $\partial D = \gamma \cup \Gamma$, where $\gamma=\left\{w=r\exp\left(\pm i\frac{\pi}{2m}\right): r\in[0,\ 1]\right\}$, is the union of the sides of the sector and $\Gamma=\left\{w=\exp(i\varphi):\varphi\in\left[-\frac{\pi}{2m},\ \frac{\pi}{2m}\right]\right\}$ is the arc of the sector; $\bar{D}$ is the closure of domain D; w' are points of arc Γ.

In the sector D we consider a stationary heat conduction problem involving heat exchange on the sides of the sector (on γ) in accordance with Newton's law with coefficient $h > 0$ and a given temperature distribution $f(w')$ on the arc Γ; this problem may be reduced [1] to the following boundary value problem for the Laplace equation:

$$\Delta T(w)=0,\ w\in D; \tag{2}$$

$$\frac{\partial}{\partial \nu}T(w)-hT(w)=0,\ w\in \operatorname{int}\gamma; \tag{3}$$

$$T(w')=f(w'),\ w'\in\Gamma; \tag{4}$$

function $f(w')$ is continuous on Γ.

In the present paper we represent the solution of problem (2)-(4) in the form of a series expressed in terms of the system of functions $\{\Omega_n(w)\}_{n=0}^{\infty}$:

$$T(w)=\sum_{n=0}^{\infty}a_n\Omega_n(w), \tag{5}$$

possessing the following properties:

a) all the Ω_n are harmonic functions in D;

b) Ω_n satisfy, for $w=r\exp\left(\pm i\frac{\pi}{2m}\right)$, $r\in(0,\ \infty)$, a homogeneous boundary-value problem of the third kind:

$$\frac{\partial}{\partial \nu}\Omega_n(w)-h\Omega_n(w)=0; \tag{6}$$

c) the set of functions $\{\Omega_n\}_{n=0}^{\infty}$ possesses the property of completeness on the arc Γ.

This method of solving problem (2)-(4) was proposed in [2]; it is close to the method presented in [3].

2. We decompose the set of functions $\{\Omega_n(w)\}_{n=0}^{\infty}$ into two subsets, one of which, $\{\Omega_{2n}\}_{n=0}^{\infty}$, is symmetric and the other, $\{\Omega_{2n-1}\}_{n=1}^{\infty}$, is antisymmetric with respect to the real axis:

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$$\Omega_{2n}(w) = \Omega_{2n}(\overline{w}),\ \Omega_{2n-1}(w) = -\Omega_{2n-1}(\overline{w}),\ w \in D. \tag{7}$$

We seek $\Omega_n(r \exp i\phi)$ in the form:

$$\Omega_{2n}(r\exp(i\varphi)) = \sum_{j=0}^{\varkappa} A_{2n}^{j} r^{\lambda_{2n}+j} \cos(\lambda_{2n}+j)\varphi, \tag{8}$$

$$\Omega_{2n-1}(r\exp(i\varphi)) = \sum_{j=0}^{\varkappa} A_{2n-1}^{j} r^{\lambda_{2n-1}+j} \sin(\lambda_{2n-1}+j)\varphi, \tag{9}$$

where we assume that κ is a positive integral parameter and λ_n and A_n are real parameters. We note that the condition $\lambda_n \geq 0$ must be satisfied, for otherwise the functions Ω_n would be unbounded in D. It may be readily verified that the functions Ω_n of the form (8), (9) are harmonic in D. Substituting functions (8) and (9) into the condition (6), we obtain relations from which κ, λ_n, and A_n can be determined:

$$\sum_{j=0}^{\varkappa} (\lambda_{2n}+j) A_{2n}^{j} r^{\lambda_{2n}+j-1} \sin(\lambda_{2n}+j)\frac{\pi}{2m} = h\sum_{j=0}^{\varkappa} A_{2n}^{j} r^{\lambda_{2n}+j} \cos(\lambda_{2n}+j)\frac{\pi}{2m}, \tag{10}$$

$$\sum_{j=0}^{\varkappa} (\lambda_{2n-1}+j) A_{2n-1}^{j} r^{\lambda_{2n-1}+j-1} \cos(\lambda_{2n-1}+j)\frac{\pi}{2m} = h\sum_{j=0}^{\varkappa} A_{2n-1}^{j} r^{\lambda_{2n-1}+j} \sin(\lambda_{2n-1}+j)\frac{\pi}{2m}. \tag{11}$$

Equating coefficients of like powers of r on the left and right sides of Eqs. (10) and (11), we obtain

$$\lambda_{2n}A_{2n}^{0} \sin\lambda_{2n}\frac{\pi}{2m} = 0,\ \lambda_{2n-1}A_{2n-1}^{0} \cos\lambda_{2n-1}\frac{\pi}{2m} = 0; \tag{12}$$

$$(\lambda_{2n}+j) A_{2n}^{j} \sin(\lambda_{2n}+j)\frac{\pi}{2m} = hA_{2n}^{j-1} \cos(\lambda_{2n}+j-1)\frac{\pi}{2m}; \tag{13}$$

$$(\lambda_{2n-1}+j) A_{2n-1}^{j} \cos(\lambda_{2n-1}+j)\frac{\pi}{2m} = hA_{2n-1}^{j-1} \sin(\lambda_{2n-1}+j-1)\frac{\pi}{2m}; \tag{14}$$

$$j = 1,\ \ldots,\ \varkappa;$$

$$hA_{2n}^{\varkappa} \cos(\lambda_{2n}+\varkappa)\frac{\pi}{2m} = 0,\ hA_{2n-1}^{\varkappa} \sin(\lambda_{2n-1}+\varkappa)\frac{\pi}{2m} = 0. \tag{15}$$

From relations (12) it follows that

$$\lambda_n = mn,\ n = 0,\ 1,\ \ldots, \tag{16}$$

and from relations (15) and (16) we obtain the equation

$$\varkappa = m. \tag{17}$$

Noting that the coefficients A_n^j, n = 0, 1, ... are determined to within an arbitrary factor, we put

$$A_n^0 = 1,\ n = 0,\ 1,\ \ldots \tag{18}$$

From expressions (13) and (14) we then establish recursion relationships for the coefficients A_n^{j+1}, j = 0, 1, ..., m − 1:

$$A_n^{j+1} = \frac{h}{mn+j+1}\ \frac{\cos j\dfrac{\pi}{2m}}{\sin(j+1)\dfrac{\pi}{2m}} A_n^j,\ n = 0,\ 1,\ \ldots \tag{19}$$

Thus, the functions $\Omega_n(w)$, n = 0, 1, ... in Eqs. (8) and (9), with parameter values from Eqs. (16)-(19), satisfy the conditions a) and b) formulated in Sec. 1.

3. We show now that the system $\{\Omega_n(w')\}_{n=0}^{\infty}$ is minimal [4] in the space $\mathscr{L}_2(\Gamma)$. In fact, in the closed disk $\overline{\mathscr{D}}$ ($\mathscr{D} = \{w = |w| < 1\}$) we consider the set of functions $\left\{\sum_{j=0}^{m} B_n^j w^{mn+j}\right\}_{n=0}^{\infty}$.

Here $B_{2n}^j = A_{2n}^j$, n = 0, 1, ...; $B_{2n-1}^j = -iA_{2n-1}^j$, n = 1, 2, ... We assume the existence of numbers $d_0, d_1, \ldots, d_N$, such that for arbitrary n_0 and $\varepsilon > 0$ the following inequality is satisfied on $\mathscr{D}$:

$$\Big|\sum_{j=0}^{m} B_{n_0}^j w^{m_0+j} - \sum_{\substack{n=0\\ n\neq n_0}}^{N} d_n \sum_{j=0}^{m} B_n^j w^{mn+j}\Big| < \varepsilon,\ w \in \partial\mathscr{D}. \tag{20}$$

We introduce the notation

$$\Phi_{n_0}(w) = \sum_{j=0}^{m} B_{n_0}^j w^{m_0+j} - \sum_{\substack{n=0\\ n\neq n_0}}^{N} d_n \sum_{j=0}^{m} B_n^j w^{mn+j} \tag{21}$$

and, at the center of the disk, we represent the function $\Phi_{n_0}(w)$, holomorphic in $\mathscr{D}$ and continuous in $\overline{\mathscr{D}}$, by means of the Cauchy formula [5]:

$$\Phi_{n_0}^{r}(0) = \frac{1}{2\pi i}\int_{\partial\mathscr{D}} \frac{\Phi_{n_0}(\exp(i\varphi))}{\exp(i\varphi)}\, d(\exp(i\varphi)). \tag{22}$$

Let m ≠ 1; differentiating both sides of Eq. (22) mn_0 and mn_0-1 times, we obtain, taking relation (20) into account, conditions on the coefficients:

$$|1 - d_{n_0-1}B_{n_0-1}^m| < \varepsilon,\ |B_{n_0-1}^{m-1} d_{n_0-1}| < \varepsilon. \tag{23}$$

If m = 1, then, differentiating the expression (22) in succession, N + 1; N, ..., 1, 0 times, we arrive at the system of inequalities

$$\begin{gathered}|d_N B_N^1| < \varepsilon,\ |d_N + d_{N-1}B_{N-1}^1| < \varepsilon,\ \ldots,\ |B_{n_0}^1 - d_{n_0+1}| < \varepsilon,\\ |1 - d_{n_0-1}B_{n_0-1}^1| < \varepsilon,\ \ldots,\ |d_1 + d_0 B_0^1| < \varepsilon,\ |d_0| < \varepsilon.\end{gathered} \tag{24}$$

Inconsistency of conditions (23) and (24) is easily verified; but this means that inequality (20) is not satisfied for $w \in \partial\mathscr{D}$. Thus, the system $\left\{\sum_{j=0}^{m} B_n^j w^{mn+j}\right\}_{n=0}^{\infty}$ is minimal in $C(\partial\mathscr{D})$, which implies minimality of $\{\Omega_n(w')\}_{n=0}^{\infty}$ in $\mathscr{L}_2(\Gamma)$.

We now prove the following

<u>Proposition I.</u> The set of functions $\{\Omega_n(w')\}_{n=0}^{\infty}$ forms a Riesz basis in $\mathscr{L}_2(\Gamma)$.

We consider the series

$$\sum_{n=0}^{\infty} \rho_n^2, \tag{25}$$

where ρ_n is given by the expressions.

$$\rho_{2n}^2 = \int_\Gamma (\cos 2nm\varphi - \Omega_{2n}(w'))^2\, d\varphi,\ n = 0,\ 1,\ \ldots, \tag{26}$$

$$\rho_{2n-1}^2 = \int_\Gamma (\sin(2n-1)m\varphi - \Omega_{2n-1}(w'))^2\, d\varphi,\ n = 1,\ 2,\ \ldots \tag{27}$$

From conditions (16) and (17) it follows that

$$\rho_n^2 = O\left(\frac{1}{n^2}\right),\ n = 0,\ 1,\ \ldots, \tag{28}$$

i.e., series (25) converges; but this means that the system $\{\Omega_n(w')\}_{n=0}^{\infty}$ is close in the mean square sense to the system $\{\cos 2nm\phi, \sin(2n+1)m\phi\}_{n=0}^{\infty}$, and is obviously a Riesz basis in $\mathscr{L}_2(\Gamma)$ [6]. Proposition 1 is a consequence of a theorem of N. K. Bari [6] relating to stability of the property of a system to form a Riesz basis for all minimal systems close to mean square.

4. We seek an approximate solution $T^K(w)$ of problem (2)-(4) as a partial sum of series (5):

$$T^K(w) = \sum_{n=0}^{K} a_n^K \Omega_n(w), \tag{29}$$

where the coefficients are determined from the condition of minimum deviation of $T^K(w')$ from $f(w')$ in the norm of the space $\mathscr{L}_2(\Gamma)$. From the results presented in [7] and also Proposition 1 it follows that for all n we have existence of the finite limit

$$\lim_{K\to\infty} a_n^K = a_n \tag{30}$$

and convergence of the sequence $\{T^K(w')\}_{K=0}^{\infty}$ on Γ as $K \to \infty$ to the function $f(w')$.

From the properties of the coefficients a_n described in [6] it follows that T(w) from Eq. (5) represents a solution of the boundary-value problem (2)-(4).

5. The method of solution proposed admits certain generalizations.

1°. Let $D_0 = \left\{w = r\exp(i\varphi) : r \in [0,\ 1],\ \varphi \in \left(0,\ \frac{\pi}{2m}\right)\right\}$ be the upper half of sector D, let D, $\Gamma_0 = \left\{w = \exp(i\varphi) : \varphi \in \left(0,\ \frac{\pi}{2m}\right)\right\}$ be the arc of its boundary, and let the function $T_\pm$ be, respectively, solutions of the following boundary-value problems:

$$\Delta T_\pm(w) = 0,\ w \in D_0; \tag{31}$$

$$\left(\frac{\partial}{\partial\nu} - h\right) T_\pm(w) = 0,\ w = r\exp(i\varphi),\ \varphi \in [0,\ 1]; \tag{32}$$

$$\mathscr{H}_\pm T_\pm(r) = 0,\ r \in [0,\ 1]; \tag{33}$$

$$T_\pm(w') = f(w'),\ w' \in \Gamma_0. \tag{34}$$

Here $\mathscr{H}_+ T_+ = \frac{\partial}{\partial\nu} T_+,\ \mathscr{H}_- T_- = T_-.$

The sets of functions $\{\Omega_{2n}(w)\}_{n=0}^{\infty}$ and $\{\Omega_{2n-1}\}_{n=1}^{\infty}$ then satisfy conditions (31) and (32), and also, as a consequence of property (7), the equations $\mathscr{H}_+\Omega_{2n}(r) = 0$, n = 0, 1, ..., and $\mathscr{H}_-\Omega_{2n-1}(r) = 0$, n = 1, 2, ... are valid. An approximate solution of problem (31)-(34) can be written in the form

$$T_\pm^K(w) = \sum_{n=0}^{K} a_{2n}^K \Omega_{2n}(w),\ T_-^K(w) = \sum_{n=1}^{K} a_{2n-1}^K \Omega_{2n-1}(w), \tag{35}$$

where the coefficients a_n^K are determined by the method of least squares.

2°. When, instead of condition (4), we are given a nonhomogeneous condition of the second or third kind on the arc Γ in problem (2)-(4),

$$\frac{\partial}{\partial r} T(w') = f(w'),\ w' \in \Gamma; \tag{36}$$

$$\frac{\partial}{\partial r} T(w') - h_\Gamma T(w') = f(w'),\ w' \in \Gamma; \tag{37}$$

$$h_\Gamma = \text{const} > 0$$

we also seek an approximate solution in the form (29).

6. Problem (2)-(4) was solved numerically for various values of the parameters m, h, and K and for a different form of function f(w') in condition (4). The controlling factor here was the mean-square error:

$$\delta(K,\ f) = \left(\int_\Gamma |f(w') - T^K(w')|^2\,|dw'|\right)^{1/2}.$$

Results of our calculations for h = 1/2, f(w') = 1, and K = 10 are shown in Figs. 1-3.

In Fig. 1 results are shown for the case m = 1 (D, a semicircular region); the error $\delta = 3.47\cdot10^{-3}$; I is the boundary of domain D; isotherms T = const for T values 0.8, 0.85, 0.9, and 0.95 correspond to curves a, b, c, and d, respectively.

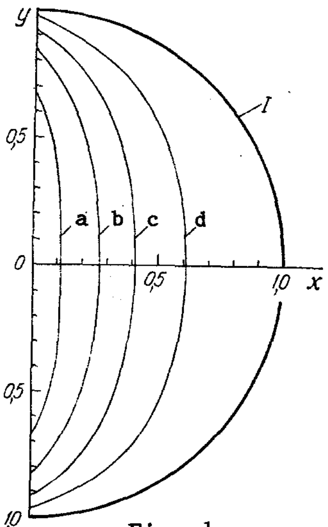

Fig. 1

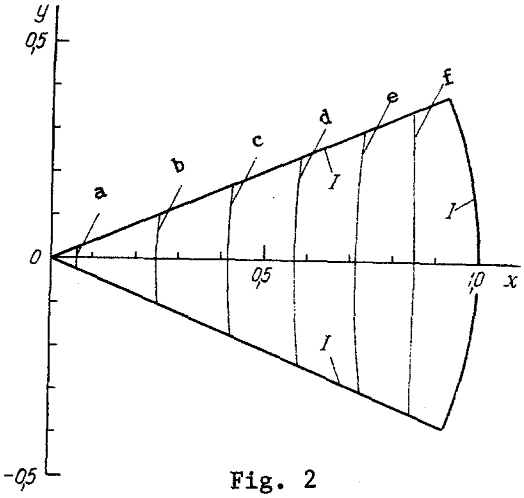

Fig. 2

Fig. 1. Isotherms for m = 1.
Fig. 2. Isotherms for m = 4.

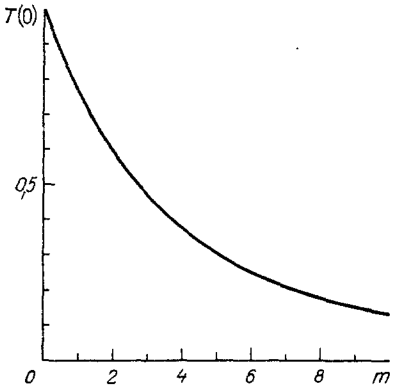

Fig. 3. Temperature at angle vertex.

In Fig. 2 results are shown for the case m = 4 (D, a sector with central angle $\pi/4$); the error $\delta = 4.32 \cdot 10^{-3}$; I is the boundary ∂D; isotherms T = const for T values 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9 correspond to curves a, b, c, d, e, and f, respectively.

Temperature T(0) at the vertex of the central angle of the sector is shown as a function of m in Fig. 3.

NOTATION

r, ϕ, polar coordinates; i, imaginary unit; $\bar{w}$, complex conjugate of w; D, circular sector; ∂D, boundary of circular sector; π/m, central angle of sector; int γ, arc γ minus endpoints; $\partial/\partial\gamma$, derivative along exterior normal to contour; Δ, Laplace operator; T, temperature; h, heat transfer coefficient; $C(\partial D)$, space of functions continuous on ∂D; $\mathcal{D} = \{w: |w| < 1\}$, unit disk; $\mathcal{L}_2(\Gamma)$, space of functions square-summable on arc Γ.

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